

# IMPROVING THE SIDE APPROXIMATION THEOREM<sup>(1,2)</sup>

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**1. Introduction.** The following version of the Side Approximation Theorem for 2-spheres is proved as Theorem 16' in [5].

**THEOREM.** *For each 2-sphere  $S$  in  $E^3$  and each  $\epsilon > 0$  there is a homeomorphism  $h: S \times [-1, 1] \rightarrow E^3$  such that*

*each  $h_t(S)$  is a polyhedral 2-sphere,*

*$D(s, h_t(s)) < \epsilon$  ( $s \in S, -1 \leq t \leq 1$ ),*

*$S$  lies except for a finite collection of mutually exclusive  $\epsilon$ -disks in  $h(S \times (-1, 1))$ ,*

*$h_{-1}(S)$  lies except for a finite collection of mutually exclusive  $\epsilon$ -disks in the interior of  $S$ , and*

*$h_1(S)$  lies except for a finite collection of mutually exclusive  $\epsilon$ -disks in the exterior of  $S$ .*

In the above we use  $D$  to represent the distance function and interior of  $S$ , exterior of  $S$  respectively to denote the bounded, unbounded components of  $E^3 - S$ . We say that  $h_1(S) = h(S \times 1)$  approximates  $S$  "almost" from the exterior of  $S$  and  $h_{-1}(S)$  approximates  $S$  "almost" from the interior of  $S$ .

In the last section of [5] the above result was generalized to open subsets of 2-spheres but the proof is complicated and should not be studied. Instead, we recommend the relatively easy treatment of the following result.

**THEOREM 1.1. SIDE APPROXIMATION THEOREM FOR 2-MANIFOLDS.** *Suppose  $M^2$  is a connected 2-manifold (perhaps noncompact) in a connected 3-manifold  $M^3$  such that*

$$M^3 - M^2 = U_1 + U_2 \text{ (mutually separated)}$$

*and  $f$  is a positive continuous function defined on  $M^2$ . Then there is a homeomorphism  $h: M^2 \times [-1, 1] \rightarrow M^3$  such that*

*(1) each  $h_t(M^2)$  is tame,*

*(2)  $D(m, h_t(m)) < f(m)$  ( $m \in M^2, -1 \leq t \leq 1$ ),*

*(3)  $M^2 - h(M^2 \times (-1, 1))$  is covered by the interiors of a locally finite*

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collection of mutually exclusive disks in  $M^2$  such that the diameter of each is less than the minimum value of  $f$  on it,

(4) for  $0 < t \leq 1$ ,  $\bar{U}_1 \cdot h_{-t}(M^2)$  is covered by the interiors of a locally finite collection of mutually exclusive disks in  $h_t(M^2)$  each of diameter less than  $f(x)$  if  $h_t(x)$  lies in the disk, and

(5) for  $0 < t \leq 1$ ,  $\bar{U}_2 \cdot h_{-t}(M^2)$  is covered by the interiors of a locally finite collection of mutually exclusive disks in  $h_{-t}(M^2)$  each of diameter less than  $f(x)$  if  $h_{-t}(x)$  lies in the disk.

In fact, if  $M^{2'}$  is a closed subset of  $M^2$  which is a tame 2-manifold with boundary,  $h$  can be chosen so that  $h_0$  is the identity on  $M^{2'}$ .

The details of the proof of Theorem 1.1 are delayed until the next section. However, the idea of the description of the homeomorphism  $h: M^2 \times [-1, 1] \rightarrow M^3$  is so simple that we outline it here. The homeomorphism  $h$  is obtained as follows.

(1) Get a locally finite collection of tame simple closed curves  $J_1, J_2, \dots$  in  $M^2$  so that each component of  $M^2 - \sum J_i$  is small.

(2) Use the approximation theorem for open subsets of 2-spheres (Theorem 8 of [2]) to get a homeomorphism  $h_0: M^2 \rightarrow M^3$  such that  $h_0 = I$  (identity) on  $M^{2'} + \sum J_i$ ,  $h_0(M^2)$  is locally tame mod  $h_0(\sum J_i)$ , and  $h_0$  does not move points far.

(3) Use a variation of Theorem 8.5 of [7] to show that  $h_0(M^2)$  is tame (and hence bicollared). Get a homeomorphism  $h: M^2 \times [-1, 1] \rightarrow M^3$  so that  $h: M^2 \times 0$  is the  $h_0$  we have already described. If the "normals"  $h(m \times [-1, 1])$  are taken small, the  $h$  satisfies the conclusion of Theorem 1.1.

An  $n$ -manifold is a metric space each of whose points has a neighborhood homeomorphic with Euclidean  $n$ -space  $E^n$ . Each point of an  $n$ -manifold with boundary has a neighborhood whose closure is a topological  $n$ -cell. Hence, an  $n$ -manifold is an  $n$ -manifold with boundary but not conversely. Since each component of an  $n$ -manifold is separable, we frequently work only with their components to have the added security of separability. If  $M$  is an  $n$ -manifold with boundary, we denote the set of its points which have neighborhoods homeomorphic with  $E^n$  by  $\text{Int } M$  (called interior  $M$ ) and denote  $M - \text{Int } M$  by  $\text{Bd } M$  (called boundary  $M$ ). For a disk  $D$  in the plane, the point set boundary is  $\text{Bd } D$  but if  $D$  is embedded in  $E^3$ , the point set boundary is more.

A 2-manifold with boundary  $M^{2'}$  is tame in a 3-manifold  $M^3$  if there is a triangulation of  $M^3$  such that  $M^{2'}$  is the closed sum of elements of the triangulation. Being such a tame manifold is a local property since a closed subset of a 3-manifold is tame if it is locally tame [1], [11]. A set  $X$  is locally tame at a point  $p$  of  $X$  if there is a neighborhood  $N$  of  $p$  and a homeomorphism of  $\bar{N}$  into a combinatorial cube that takes  $\bar{N} \cdot X$  onto a polyhedron. A set is locally tame if it is locally tame at each of its points.

In general, a set is *tame* in a manifold if the set is a geometric complex of some combinatorial triangulation (perhaps curvilinear) of the manifold. If the manifold already has a combinatorial structure we insist that the combinatorial triangulation that makes the tame set a geometric complex is isomorphic to a subdivision of the original combinatorial triangulation. Hence, a set  $X$  in a combinatorial manifold is tame if there is a homeomorphism of the manifold onto itself that takes  $X$  onto a polyhedron.

A *Sierpiński curve* is a set homeomorphic to the set obtained by removing from a 2-sphere the interiors of a null sequence of mutually exclusive subdisks whose sum is dense in the 2-sphere. A *null* sequence is one which for each positive number  $\epsilon$  has at most a finite number of elements with diameters more than  $\epsilon$ . A Sierpiński curve in a 3-manifold is called *tame* if it lies on some tame disk in the 3-manifold.

**2. Proof of Theorem 1.1.** Let  $p_0, q_0$  be points in  $U_1, U_2$  respectively and  $\epsilon$  be a positive number such that no  $2\epsilon$ -subset of  $V(M^2, \epsilon)$  separates  $p_0$  from  $q_0$  in  $M^3$ . We use  $V(A, \epsilon)$  to denote the set of all points whose distance from  $A$  is less than  $\epsilon$ . We suppose that  $f < \epsilon$  and so small that for each point  $p$  of  $M^2$ ,  $V(p, 2f(p))$  lies in an open 3-cell in  $M^3$ . The reason for making  $f$  so small is to make it easy to show that  $h_1(M^2)$  and  $h_{-1}(M^2)$  lie except for small holes in different ones of  $U_1, U_2$ . See Theorem 5.2 of §5.

We use  $\min_f(X)$  to denote the minimum value of  $f$  on  $X$ .

The proof is broken into four pieces—the first three following our outline in the introduction and the fourth showing that the homeomorphism  $h$  we describe satisfies the conditions of the theorem.

**1. Chopping  $M^2$  into small pieces.** It is known that  $M^2$  can be triangulated into small simplexes. It follows further from Theorem 5 of [4] that there is a locally finite collection  $\{D_i\}$  of disks in  $M^2$  such that

$\{\text{Int } D_i\}$  covers  $M^2$  but no subcollection does,

diameter  $D_i < \min_f(D_i)/2$ ,

for each  $D_i$  there is an open 3-cell  $O_i$  in  $M^3$  and a 2-sphere  $S_i$  so that  $D_i \subset S_i \subset O_i$ .

Let  $\{D'_i\}$  be a collection of disks such that

$$D'_i \subset \text{Int } D_i \quad \text{and} \quad \{\text{Int } D'_i\} \text{ covers } M^2.$$

It follows from Theorem 1 of [6] that for each positive integer  $i$  there is a tame Sierpiński curve  $X_i$  in  $S_i$  such that each component of  $S_i - X_i$  is of diameter less than  $D(\text{Bd } D_i, \text{Bd } D'_i)/2$ . Let  $J_i$  be a simple closed curve in  $X_i \cdot (\text{Int } D_i - D'_i)$  that separates  $D'_i$  from  $\text{Bd } D_i$  in  $D_i$ . To find such a  $J_i$ , let  $g_i$  be a map of  $S_i$  onto a 2-sphere  $g_i(S_i)$  such that the inverse of each point of  $g_i(S_i)$  is either the closure of a component of  $S_i - X_i$  or a point of  $X_i$  not on such a closure. Let  $J'_i$  be a simple closed curve on  $g_i(S_i)$  that separates  $g_i(D'_i)$  from  $g_i(S_i - \text{Int } D_i)$  and which misses each of the

countable set of points of  $g_i(S_i)$  which have nondegenerate inverses. Then  $J_i = g_i^{-1}(J'_i)$ . Denote the disk in  $D_i$  bounded by  $J_i$  by  $D'_i$ . Then  $D'_i \subset \text{Int } D'_i \subset D'_i \subset \text{Int } D_i$ . Also,  $J_i = \text{Bd } D'_i$  is tame.

2. *Defining  $h_0$ .* We first get a function  $f'$  which tells us how close to approximate  $M^2$ . Let  $f'$  be a continuous function defined on  $M^2$  such that for each element  $D_i$  of  $\{D_i\}$ ,

$$p \in D_i \Rightarrow f'(p) < \min_f(D_i)/4,$$

$$p \notin \text{Int } D_i \Rightarrow D(p, D'_i) > f'(p), \text{ and}$$

$$D(p, q) < f'(q) \Rightarrow f(p) < 2f(q).$$

Let  $U$  be an open set containing  $M^2 - (M^{2'} + \sum J_i)$  such that each component of  $U$  lies in a 3-cell in  $M^3$  and no component of  $U$  contains two components of  $M^2 - (M^{2'} + \sum J_i)$ . It follows from Theorem 7 of [2] that there is a homeomorphism  $h_0$  of  $M^2$  into  $M^3$  such that

$$h_0(M^2) \text{ is locally tame at } h_0(p) \text{ if } p \in M^2 - (\text{Bd } M^{2'} + \sum J_i),$$

$$D(m, h_0(m)) < f'(m)/2 < f(m)/8,$$

$$h_0 = I \text{ (identity) on } M^{2'} + \sum J_i,$$

and  $h_0$  is so close to the identity that there is a homotopy  $g: M^2 \times [0, 1] \rightarrow M^3$  such that

$$g_0 = h_0,$$

$$g_1 = I,$$

$$g_i = I \text{ on } M^{2'} + \sum J_i, \text{ and}$$

$$g_i(M^2 \cdot U) \subset U.$$

Although Theorem 7 of [2] says nothing about a homotopy  $g$ , it does imply that  $h_0$  can be taken close to the identity. To see that taking  $h_0$  near the identity implies a  $g$ , consider a locally finite collection of topological cubes such that each component  $u$  of  $U$  lies in one of them (say  $C_u$ ). Let  $h_u$  be a homeomorphism of  $C_u$  onto a canonical cube. Then there is a homotopy  $g$  if we take  $h_0$  close enough to the identity that for each  $u$  and for each point  $p \in M^2 \cap u$ , the point  $h_0(p)$  lies in  $C_u$  and the segment from  $h_u(p)$  to  $h_u h_0(p)$  lies in  $h_u(u)$ .

Let  $h'_i$  be the homeomorphism that is  $h_0$  on  $\sum_i D'_i$  and the identity on  $M^2 - \sum_i D'_i$ . It follows from Corollary 7.2 of §7 that each  $h'_i(M^2)$  separates  $p$  from  $q$  in  $M^3$ . Hence  $h_0(M^2)$  is two sided in  $M^3$ .

3. *Defining  $h$ .* It follows from a modification of Theorem 8.5 of [7] that  $h_0(M^2)$  is locally tame. This modification is spelled out in Theorem 3.1 of the next section. Hence  $h_0(M^2)$  locally has a cartesian product neighborhood and it follows from [8] that there is a homeomorphism  $h: M^2$

$\times [-1, 1] \rightarrow M^3$  such that  $h_0$  is the homeomorphism previously defined and each  $h_i(M^2)$  is locally tame. Suppose that for each point  $p$  of  $M^2$ , diameter  $h(p \times [-1, 1]) < f'(p)/2$ . This implies that

$$D(p, h_i(p)) < f'(p) < f(p)/4.$$

Except for a possible exchange of  $t$  and  $-t$  this homeomorphism  $h$  satisfies the conclusion of Theorem 1.1.

4. *Showing that  $h$  satisfies Conditions 2, 3, 4, 5.* Condition 2 of the conclusion of Theorem 1 is satisfied because

$$D(m, h_0(m)) < f(m)/8 \text{ and}$$

$$\text{diameter } h(m \times [-1, 1]) < f(m)/8.$$

Each component of  $M^2 - h(M^2 \times (-1, 1))$  lies on the interior of an element of  $\{D_i\}$  since

$$\sum J_i \subset h_0(M^2) \subset h(M^2 \times (-1, 1)) \text{ and}$$

each component of  $M^2 - \sum J_i$  lies in a disk  $D'_i$  of  $\{D'_i\}$ .

It follows from Theorem 4.1 that there is a locally finite collection of mutually exclusive disks  $G_1, G_2, \dots$  in  $M^2$  such that the interiors of the  $G_i$ 's cover  $M^2 - h(M^2 \times (-1, 1))$ . The  $G_i$ 's are small since if  $G_i \subset D_j$ ,

$$\text{diameter } G_i \leq \text{diameter } D_j < \min_f(D_j)/2 \leq \min_f(G_i)/2.$$

Hence Condition 3 is satisfied.

As a step toward showing that Conditions 4 and 5 are satisfied we show that each component  $A$  of  $M^2 \cdot h_1(M^2)$  lies in an  $h_1(\text{Int } D_i)$  for some element  $D_i$  of  $\{D_i\}$ . Since  $\sum J_i \cdot h_1(M^2) = 0$ ,  $A$  lies in a component of  $M^2 - \sum J_i$  and hence in a  $D'_i$ . Then  $A$  lies in the corresponding  $h_1(\text{Int } D_i)$  because  $\text{Cl}(h_1(M^2 - D_i))$  misses  $D'_i$  as can be seen from the facts that  $D(p, h_i(p)) < f'(p)$  and  $f'(q) > D(q, D'_i)$  if  $q \notin \text{Int } D_i$ .

The  $h_1(D_i)$ 's give a locally finite collection of disks whose interiors cover  $M^2 \cdot h_1(M^2)$ . It follows from Theorem 4.1 that there is a locally finite collection of mutually exclusive disks  $E_1^1, E_2^1, \dots$  in  $h_1(M^2)$  whose interiors cover  $M^2 \cdot h_1(M^2)$  such that each of these disks lies in an  $h_1(D_i)$ . If  $E_i^1 \subset h_1(D_j)$ , then

$$\begin{aligned} \text{diameter } E_i^1 &< \text{diameter } h_1(D_j) \leq \text{diameter } D_j + 2 \min_f(D_j)/4 \\ &\leq \min_f(D_j) \leq \min_f h^{-1}(E_i^1). \end{aligned}$$

Since  $h_1(M^2) - \sum E_i^1$  is connected, it lies in one of  $U_1, U_2$ —so with a possible exchange of  $U_1, U_2$ , Condition 4 is satisfied for  $t = 1$ .

Similarly, it follows that for  $-1 \leq t < 0$  or  $0 < t \leq 1$  there is a collection of mutually exclusive small disks  $E_1^t, E_2^t, \dots$  in  $h_t(M^2)$  such that  $h_t(M^2) - \sum E_i^t$  lies in one of  $U_1, U_2$ . If  $t, t'$  are of the same sign,  $h_t(M^2) - \sum E_i^t$

and  $h_r(M^2) - \sum E_i^r$  lie in the same one of  $U_1, U_2$  or else there is a  $t_0 \neq 0$  and a sequence  $t_1, t_2, \dots$  converging to  $t_0$  such that  $h_{t_0}(M^2) - \sum E_i^{t_0}$  and  $h_{t_j}(M^2) - \sum E_i^{t_j}$  ( $j > 0$ ) lie in different ones of  $U_1, U_2$ . This is impossible since for  $j$  sufficiently large,  $h_{t_j}\{h_{t_0}^{-1}[h_{t_0}(M^2) - \sum E_i^{t_0}]\}$  lies in the one of  $U_1, U_2$  containing  $h_{t_0}(M^2) - \sum E_i^{t_0}$  and there would not be a finite collection of small disks in  $h_{t_j}(M^2)$  covering  $h_{t_j}\{h_{t_0}^{-1}[h_{t_0}(M^2) - \sum E_i^{t_0}]\}$ .

Condition 5 is established when we show that it is not the same one of  $U_1, U_2$  containing  $h_1(M^2) - \sum E_i^1$  and  $h_{-1}(M^2) - \sum E_i^{-1}$ . This follows from Theorem 5.2 of §5. The only difficult condition to check in applying Theorem 5.2 is to see that Condition 5 of the hypothesis of that theorem is satisfied. We do this in the next paragraph.

Suppose  $G_k$  is one of the disks of Condition 3 that intersects the disk  $E_i$  of Condition 4 in a point  $p$  and  $h_1(q) = p$ . We show that  $G_k + E_i$  lies in an open 3-cell in  $M^3$  by showing that each lies in  $V(q, 2f(q))$ . First, there is a  $D_j$  such that  $E_i \subset h_1(D_j)$ . Then since  $h_1$  does not move  $x$  farther than  $f'(x)$ ,

$$\begin{aligned} E_i \subset h_1(D_j) &\subset V(D_j, \min_f(D_j)/4) \subset V(q, \text{diameter } D_j + \min_f(D_j)/4) \\ &\subset V(q, \min_f(D_j)/2 + \min_f(D_j)/4) \subset V(q, f(q)). \end{aligned}$$

Also, since  $D(p, q) < f'(q), f(p) < 2f(q)$  and

$$\begin{aligned} G_k &\subset V(p, f(p)/2) \subset V(p, f(q)) \subset V(q, D(p, q) + f(q)) \\ &\subset V(q, f'(q) + f(q)) \subset V(q, f(q)/4 + f(q)) \subset V(q, 2f(q)). \end{aligned}$$

Using the fact that any homeomorphism of a triangulated 3-manifold with boundary into a triangulated 3-manifold can be approximated with a piecewise linear homeomorphism as shown by Theorem 9 of [3] and Theorem 2 of [10], we obtain the following variation of Theorem 1.1.

**THEOREM 1.1'.** *Suppose  $M^2$  is a connected polyhedral 2-manifold,  $h'$  is a homeomorphism of  $M^2$  into a triangulated 3-manifold  $M^3$  such that  $h'(M^2)$  separates  $M^3$ , and  $f$  is a positive continuous function defined on  $M^2$ . Then there are three locally finite collections  $\{G_i\}, \{E_i\}, \{F_i\}$  of mutually exclusive disks in  $M^2$  and a piecewise linear homeomorphism  $h: M^2 \times [-1, 1] \rightarrow M^3$  such that*

$$D(h'(m), h_t(m)) < f(m) \quad (m \in M^2, -1 \leq t \leq 1),$$

$$\text{diameter } h'(G_i) < \min_f(G_i),$$

$$\text{diameter } h_{-1}(E_i) < \min_f(E_i),$$

$$\text{diameter } h_1(F_i) < \min_f(F_i),$$

$$h'(M^2 - \sum \text{Int } G_i) \subset h(M^2 \times (-1, 1)),$$

and

$h'(M^2)$  separates  $h_{-1}(M^2 - \sum \text{Int } E_i)$  from  $h_1(M^2 - \sum \text{Int } F_i)$  in  $M^3$ .

In fact, if  $P$  is a polyhedron in  $M^2$  on which  $h'$  is piecewise linear,  $h$  can be chosen so that  $h_0 = h'$  on  $P$ .

**3. Tame mod tame sets.** It was shown in [7] that a 2-sphere in  $E^3$  is tame if it is locally tame mod the sum of a finite number of sets each of which is either a tame arc or a tame Sierpiński Curve. (This result was used in showing that the 2-manifold  $h_0(M^2)$  considered in the preceding two sections was locally tame.) The following is a mild extension of Theorem 8.5 of [7].

**THEOREM 3.1.** *Suppose  $M^2$  is a 2-manifold embedded in a 3-manifold  $M^3$ ,  $\{X_i\}$  is a countable collection each of whose elements is either a tame arc or a tame Sierpiński curve in  $M^2$ ,  $U$  is an open subset of  $M^2$  such that  $M^2$  is locally tame at each point of  $U - \sum X_i$ . Then  $M^2$  is locally tame at each point of  $U$ .*

**Proof.** We first prove the theorem in the case where there is only one element  $X_1$  in  $\{X_i\}$ . Let  $p$  be a point of  $X \cdot U$  under consideration,  $D$  a disk in  $U$ ,  $O^3$  an open 3-cell, and  $S$  a 2-sphere, such that  $p \in \text{Int } D \subset S \subset O^3 \subset M^3$ . Then there is an arc or Sierpiński curve  $X'_1$  in  $X_1 \cdot D$  such that for some open subset  $N$  of  $\text{Int } D$   $p \in N \cdot X_1 \subset X'_1$ . It follows from Theorem 8.5 of [7]  $S$  is locally tame at each point of  $N$ . Since  $p \in N \subset M^2$ ,  $M^2$  is locally tame at  $p$ .

Now that we have disposed of the special case, let  $Y$  be the set of all points of  $U$  at which  $M^2$  is not locally tame. Then  $Y$  is a relatively closed subset of  $M^2$  which lies in  $\sum X_i$ . It follows from the Baire Category Theorem that there is a point  $q$  of  $Y$  and an open subset  $N$  of  $U$  such that  $q \in N$  and a single  $X_i$  contains  $Y \cdot N$ . It follows from the special case treated in the preceding paragraph that  $M^2$  is locally tame at each point of  $N$ . But then  $M^2$  is locally tame at  $q$ .

**4. Building disks about sets.** In the proof of Theorem 1.1 we showed that  $h_1(M^2)$  lay except for a set with small components in  $M^3 - M^2$ . In this section we point out why this implies that  $h_1(M^2)$  lies except for a locally finite collection of mutually exclusive small disks on one side of  $M^2$ .

**THEOREM 4.1.** *Suppose  $M^2$  is a connected 2-manifold,  $X$  is a closed subset of  $M^2$ , and  $\{D_i\}$  is a locally finite collection of disks in  $M^2$  such that each component of  $X$  lies on the interior of one of the disks. Then there is a locally finite collection of mutually exclusive disks  $E_1, E_2, \dots$  in  $M^2$  such that each component of  $X$  lies on the interior of an  $E_i$  and each  $E_i$  lies in some  $\text{Int } D_j$ .*

**Proof.** If  $C$  is a component of  $X$  in  $\text{Int } D_1$ , there is a disk  $D_C$  such that  $C \subset \text{Int } D_C \subset D_1$  and  $X \cdot \text{Bd } D_C = 0$ . By so covering each component of  $X$  not in any  $\text{Int } D_2, \text{Int } D_3, \dots$ , we find that there is a finite collection of

disks  $D_{11}, D_{12}, \dots, D_{1n_1}$  in  $D_1$  such that each  $X \cdot \text{Bd } D_{1i} = 0$ , and each component of  $X$  lies in the interior of one of  $D_{11}, D_{12}, \dots, D_{1n_1}, D_2, D_3, \dots$ .

We next replace  $D_2$  with a finite collection of disks  $D_{21}, D_{22}, \dots, D_{2n_2}$  in  $D_2$  such that each  $X \cdot \text{Bd } D_{2i} = 0$  and each component of  $X$  lies in the interior of one of  $D_{11}, D_{12}, \dots, D_{1n_1}, D_{21}, D_{22}, \dots, D_{2n_2}, D_3, D_4, \dots$ . Continuing in this fashion we get a locally finite collection of  $D_{ij}$ 's such that each  $D_{ij}$  lies in  $D_i$ , each  $X \cdot \text{Bd } D_{ij} = 0$ , and each component of  $X$  lies in some  $\text{Int } D_{ij}$ .

Throw away each  $D_{ij}$  that lies in a larger one. This causes the closure of each component of  $D_{ij} - \sum \text{Bd } D_n$  to be a disk rather than an annulus or something worse. The collection of closures of such components may not be a locally finite collection but the subcollection of those which intersect  $X$  is a locally finite collection. If two of these disks intersect, the intersection is on the boundary so if each is shrunk slightly but not enough to uncover any point of  $X$  or even bring it to the boundary, we obtain a collection of disks satisfying the conditions of the theorem.

The following result follows from Theorem 4.1 and the fact that a connected 2-manifold minus the sum of a locally finite collection of mutually exclusive disks is connected.

**COROLLARY 4.2.** *Under the hypotheses of Theorem 4.1 there is a component  $U$  of  $M^2 - X$  and a locally finite collection of mutually exclusive disks  $E_1, E_2, \dots$  in  $M^2$  such that each  $E_i$  lies in a  $D_j$  and the  $E_i$ 's cover  $M^2 - U$ .*

**EXAMPLE.** Corollary 4.2 is not true if we weaken the hypothesis by supposing that each component of  $X$  lies in some  $D_i$  rather than in some  $\text{Int } D_i$ . One could let  $M^2$  be a 2-sphere,  $D_1$  and  $D_2$  be two disks in  $M^2$  with a common boundary, and  $X$  be the sum of  $D_1$  and a sequence of mutually exclusive simple closed curves in  $\text{Int } D_2$  converging homeomorphically to  $\text{Bd } D_1 = \text{Bd } D_2$ .

The above is essentially the only counterexample to the modified Corollary 4.2 as may be seen by the following theorems which are not used elsewhere in this paper but included since they show an extent to which Theorem 4.1 and Corollary 4.2 can be strengthened.

**THEOREM 4.3.** *Suppose  $M^2$  is a 2-manifold,  $\{D_i\}$  is a locally finite collection of disks in  $M^2$ , and  $X$  is a closed subset of  $M^2$  such that each component of  $X$  lies in a  $D_i$ . Then  $M - X$  has at least one component which does not lie in any  $D_i$  unless possibly  $M^2$  is a 2-sphere which is the sum of two  $D_i$ 's.*

**Proof.** Suppose each component of  $M^2 - X$  lies in some  $D_i$ . We show that under this condition there are two  $D_i$ 's whose sum is  $M^2$ .

Let  $U_0$  be a component of  $M^2 - X$  and  $D_1, D_2, \dots, D_n$  be the  $D_i$ 's containing  $U_0$ . Let  $Y_i$  be the set of all points  $p$  of  $D_i$  such that each arc in



$M^2$  from  $p$  to  $\text{Bd } D_i$  intersects  $X$ , and  $C_i$  be the component of  $Y_i$  containing  $U_0$ . Denote the point set boundary in  $M^2$  of  $C_i$  by  $F_i$ —that is,  $F_i = C_i \cdot (\overline{M^2 - C_i})$ . Then  $F_i$  is connected and lies in  $X$ .

Consider the case where there is a  $D_j$  containing an  $F_i$  such that  $C_i \not\subset D_j$ . We show that under this condition that  $D_i + D_j = M^2$ . Suppose not. Let  $q_0 \in C_i - D_j$ . There would be a disk  $E_j$  in  $M^2$  slightly larger than  $D_j$  such that  $D_j \subset \text{Int } E_j$ ,  $q_0 \notin E_j$ , and  $D_i + E_j \neq M^2$ . Since  $F_i \subset \text{Int } E_j$ , there is a disk  $E_i \subset C_i$  such that  $q_0 \in \text{Int } E_i$  and  $\text{Bd } E_i \subset E_j$ . Let  $E'_j$  be the disk in  $E_j$  bounded by  $\text{Bd } E_i$ . The disks  $E_i$ ,  $E'_j$  have the same boundary but not the same interiors since  $q_0 \in \text{Int } E_i$ ,  $q_0 \notin \text{Int } E'_j$ . Hence  $E_i + E'_j$  is a 2-sphere. This is contrary to the condition that  $D_i + E_j \neq M^2$  implied by the false assumption that  $D_i + D_j \neq M^2$ . We suppose henceforth that  $F_i \subset D_j$  implies that  $C_i \subset D_j$ .

Since there are only a finite number of  $C_i$ 's, we pick one of these (say  $C_1$ ) such that it ( $C_1$ ) is not properly contained in any other  $C_i$ . Then  $C_1$  contains the component of  $X$  containing  $F_1$  or else there is a larger  $C_i$  and  $C_1$  is a component of  $C_1 + X$ . Let  $G_1, G_2, \dots$  be a decreasing sequence of disks in  $M^2$  such that  $G_{i+1} \subset \text{Int } G_i$ ,  $G_1 \cdot G_2 \cdot \dots = C_1$ , and  $X \cdot \text{Bd } G_i = 0$ . Let  $U_i$  be the component of  $M^2 - X$  containing  $\text{Bd } G_i$ . It follows from the local finiteness of  $\{D_i\}$  that there is one of the  $D_i$ 's (say  $D_j$ ) such that  $D_j$  contains infinitely many of the  $U_i$ 's. For convenience we suppose  $D_j$  contains all the  $U_i$ 's. Unless  $D_1 + D_j$  covers  $M^2$ , there is an integer  $k$  such that  $G_k + D_j$  does not cover  $M^2$ . Since  $\text{Bd } G_k$  bounds a disk in  $D_j$ , this implies that  $G_k \subset D_j$ . But since  $U_k$  lies in  $D_j$ , each arc in  $M^2$  from  $\text{Bd } G_k$  to  $\text{Bd } D_j$  intersects  $X$ . Also, each arc in  $M^2$  from  $G_k$  to  $\text{Bd } D_j$  intersects  $X$  and  $C_1$  is not as large as supposed—it should have contained  $G_k$ .

**EXAMPLE.** Theorem 4.3 is not true if we do not insist that  $\{D\}$  is locally finite as can be seen by letting  $M^2$  be the plane,  $D_i$  be the round disk with center at the origin and radius  $i$ , and  $X = \sum \text{Bd } D_i$ . However, the following result shows that even without local finiteness on  $\{D_i\}$ , there cannot be two large components of  $M^2 - X$ .

**THEOREM 4.4.** *Suppose  $M^2$  is a connected 2-manifold,  $\{D_i\}$  is a collection of disks in  $M^2$ ,  $X$  is a closed subset of  $M^2$  such that each component of  $X$  lies in some  $D_i$ , and  $U$  is a component of  $M^2 - X$  that does not lie in any  $D_i$ . Then each component of  $M^2 - U$  lies in some  $\{D_i\}$ .*

**Proof.** Suppose the component  $C$  of  $M^2 - U$  fails to lie in any  $D_i$ . Let  $p$  be a point of  $C$  such that each neighborhood of  $p$  intersects  $U$ . Let  $C'$  be the component of  $X$  containing  $p$ . Then  $C'$  lies in some  $D_i$  (say  $D_1$ ). Since  $D_1$  does not contain  $C$ , there is a disk  $E$  in  $M^2$  such that  $C' \subset D_1 \subset \text{Int } E$  but  $C \not\subset E$  and  $U \not\subset E$ . Since the component of  $X \cdot E$  containing  $C'$  does not intersect  $\text{Bd } E$ , there is a disk  $E'$  in  $E$  such that  $C' \subset \text{Int } E'$

and  $X \cdot \text{Bd } E' = 0$ . Then  $\text{Bd } E' \subset U$  and  $C \subset E' \subset E$ . The assumption that the component  $C$  of  $M^2 - U$  fails to lie in any  $D_i$  led to the contradiction that  $C \subset E$  and  $C \not\subset E$ .

**5. Separating 3-manifolds with disks.** If  $S^2$  is a 2-sphere in  $E^3$ ,  $h$  a homeomorphism of  $S^2 \times [-1, 1] \rightarrow E^3$  such that  $S^2 \subset h(S^2 \times (-1, 1))$ , then one of  $h_{-1}(S^2)$ ,  $h_1(S^2)$  lies in the bounded component of  $E^3 - S^2$  and the other lies in the unbounded component unless  $h(S^2 \times (-1, 1))$  contains the bounded component of  $E^3 - S^2$ . This result follows from the Invariance of Domain Theorem. Something can be said about the matter in the case  $h(S^2 \times (-1, 1))$  contains all of  $S^2$  except for some small holes as evidenced by Theorem 15 of [5]. In this section we generalize Theorems 14, 15 of [5] to see what can be concluded about  $M^2$  in  $M^3$ .

**THEOREM 5.1.** *Suppose  $\{A_i\}$ ,  $\{B_i\}$  are locally finite collections of mutually exclusive disks in a 3-manifold  $M^3$  such that if an element  $A_r$  of  $\{A_i\}$  intersects an element  $B_s$  of  $\{B_i\}$ , then  $A_r + B_s$  lie in an open 3-cell in  $M^3$ . Then if  $\sum A_i + \sum B_i$  separates two points  $p, q$  in  $M^3$ , there are elements  $A_u$ ,  $B_v$  in  $\{A_i\}$ ,  $\{B_i\}$  respectively such that  $A_u \cdot B_v \neq 0$  and  $A_u + B_v$  separates  $p$  from  $q$  in  $M^3$ .*

**Proof.** This theorem is an extension of Theorem 14 of [5] and is proved the same way. We suppose that  $\sum B_i + \sum_{i \geq 2} A_i$  does not separate  $p$  from  $q$  but  $\sum A_i + \sum B_i$  does. Let  $pq$  be an arc from  $p$  to  $q$  in  $\sum B_i + \sum_{i \geq 2} A_i$  which intersects  $A_1$  as few times as possible. As pointed out in the proof of Theorem 14 of [5],  $pq$  intersects  $A_1$  in only a finite number of points and pierces it at each point at which it intersects it. Also, as pointed out in that same proof, there is an element  $B_i$  of  $\{B_i\}$  and a component  $U$  of  $A_1 - B_i$  such that  $U \cdot \text{Bd } A_1 = 0$  and  $U \cdot pq$  is precisely one point.

Let  $O^3$  be an open 3-cell in  $M^3$  containing  $A_1 + B_i$ . If  $A_1 + B_i$  does not separate  $p$  from  $q$ , there is a simple closed curve  $J'$  in  $M^3$  such that  $pq \subset J'$  and  $(J' - pq) \cdot (A_1 + B_i) = 0$ . Adjust  $J'$  near  $M^3 - O^3$  to get a simple closed curve  $J$  in  $O^3$  such that for some neighborhood  $N$  of  $A_1 + B_i$ ,  $J' \cdot N = J \cdot N$ . However, this violates Theorem 13 of [5]. Hence, the assumption that  $A_1 + B_i$  does not separate  $p$  from  $q$  is false.

**THEOREM 5.2.** *Suppose*

*$M^2$  is a connected 2-manifold (perhaps noncompact) in a connected 3-manifold  $M^3$ ,*

*$p, q$  are points in different components of  $M^3 - M^2$ ,*

*$\epsilon$  is a positive number such that no  $2\epsilon$ -subset of  $V(M^2, \epsilon)$  separates  $p$  from  $q$  in  $M^3$ , and*

*$h$  is a homeomorphism of  $M^2 \times [-1, 1] \rightarrow M^3$  such that*

*(1)  $h(M^2 \times [-1, 1]) \subset V(M^2, \epsilon)$ ,*

*(2)  $M^2$  contains a locally finite collection  $\{D_i\}$  of mutually exclusive  $\epsilon$ -disks*

such that  $M^2 - \sum D_i \subset h(M^2 \times (-1, 1))$ ,

(3)  $h_{-1}(M^2)$  contains a locally finite collection  $\{E_i\}$  of mutually exclusive  $\epsilon$ -disks such that  $M^2 \cdot h_{-1}(M^2) \subset \sum E_i$ ,

(4)  $h_1(M^2)$  contains a locally finite collection  $\{F_i\}$  of mutually exclusive  $\epsilon$ -disks such that  $M^2 \cdot h_1(M^2) \subset \sum F_i$ , and

(5) if an element  $D$  of  $\{D_i\}$  intersects an element  $G$  of  $\{E_i\} + \{F_i\}$ , then  $D + G$  lies in an open 3-cell in  $M^3$ .

Then  $h_{-1}(M^2) - \sum E_i$  lies in one component of  $M^3 - M^2$  and  $h_1(M^2) - \sum F_i$  lies in the other.

**Proof.** This theorem is an analogue of Theorem 15 of [5] and is proved in the same way. We note that  $\{E_i\} + \{F_i\}$  is a locally finite collection of mutually exclusive disks and  $\{D_i\}$  is another such collection. It follows from Theorem 5.1 that there is an arc from  $p$  to  $q$  in  $M^3 - (\sum D_i + \sum E_i + \sum F_i)$ . The first point of  $M^2 + h(M^2 \times [-1, 1])$  on this arc in the order from  $p$  to  $q$  belongs to one of  $h_{-1}(M^2) - \sum E_i$ ,  $h_1(M^2) - \sum F_i$  and the last such point belongs to the other.

**6. Triangulations with tame skeletons.** One might consider our proof of Theorem 1.1 neater if instead of chopping  $M^2$  up helter-skelter by the  $J_i$ 's we had taken a triangulation of it with a tame 1-skeleton. In this section we show that any 2-manifold  $M^2$  embedded in a 3-manifold can be given a fine triangulation with a tame 1-skeleton. The 1-skeleton is picked in a certain tame Sierpiński-like set to ensure that it is tame.

**THEOREM 6.1.** Suppose  $M^2$  is a connected 2-manifold embedded in a 3-manifold  $M^3$  and  $f$  is a positive continuous function defined on  $M^2$ . Then there is a null sequence of mutually exclusive disks  $E_1, E_2, \dots$  in  $M^2$  such that

$$\text{diameter } E_i < \min_j E_j,$$

$$\sum E_i \text{ is dense in } M_2, \text{ and}$$

$$M^2 - \sum \text{Int } E_i \text{ lies in a tame 2-manifold in } M^3.$$

**Proof.** As in the proof of Theorem 1.1 we let  $\{D'_i\}$  be a locally finite collection of disks in  $M^2$  such that

$$\{\text{Int } D'_i\} \text{ covers } M^2 \text{ but no subcollection does,}$$

$$\text{Bd } D'_i \text{ is tame,}$$

$$\text{diameter } D'_i < \min_j D'_j, \text{ and}$$

$$\text{each } D'_i \text{ lies on a 2-sphere in an open 3-cell in } M^3.$$

The closure of each component of  $M^2 - \sum \text{Bd } D'_i$  is a disk with a tame boundary. Let  $F_1, F_2, \dots$  be the collection of these closures and  $S_i, O_i$  be a 2-sphere and open 3-cell respectively such that  $F_i \subset S_i \subset O_i \subset M^3$ .

It follows from Theorem 9.1 of [7] that there is a tame Sierpiński curve  $X'_i$  in  $S_i$  such that each component of  $S_i - X'_i$  is of diameter less than  $1/i$  and  $\text{Bd } F_i$  belongs to the set of inaccessible points of  $X'_i$ . Then  $X_i = X'_i \cdot F_i$  is a tame Sierpiński curve in  $F_i$  that contains  $\text{Bd } F_i$ .

The  $E_i$ 's are chosen so that  $M^2 - \sum \text{Int } E_i = \sum \text{Bd } F_i + \sum X_i$ . It follows from the Approximation Theorem for Surfaces (Theorem 7 of [2]) that there is a homeomorphism  $h$  of  $M^2$  into  $M^3$  such that  $h$  is the identity on  $\sum \text{Bd } F_i + \sum X_i$  and  $h(M^2)$  is locally tame off  $\sum \text{Bd } F_i + \sum X_i$ . It follows from Theorem 3.1 that  $h(M^2)$  is tame.

**THEOREM 6.2.** *The  $E_i$ 's of Theorem 6.1 may be chosen so that there is a monotone map  $g$  of  $M^2$  onto itself such that for each point  $m$  of  $M^2$*

$$D(m, g(m)) < f(m) \text{ and}$$

$$g^{-1}(m) \text{ is either an } E_i \text{ or a point of } M^2 - \sum E_i.$$

**Proof.** It follows from [12] that there is a monotone map  $g_i$  of  $F_i$  onto itself that is the identity on  $\text{Bd } F_i$  and such that for each point  $x$  of  $F_i$ ,  $g_i^{-1}(x)$  is either a point of  $F_i - \sum E_j$  or an  $E_j$  in  $F_i$ . The map  $g$  required by Theorem 6.2 is  $g_i$  on each  $F_i$  and the identity elsewhere.

**THEOREM 6.3.** *Suppose  $M^2$  is a 2-manifold embedded in a 3-manifold and  $f$  is a positive continuous function defined on  $M^2$ . Then there is a triangulation  $T$  of  $M^2$  such that the 1-skeleton of  $T$  is tame and for each simplex  $\sigma$  of  $T$ , diameter  $\sigma < \min_i f_i(\sigma)$ .*

**Proof.** Since we can operate on the components of  $M^2$  one at a time, we suppose with no loss of generality that  $M^2$  is connected.

Let  $f_1$  be a positive continuous function defined on  $M^2$  such that

$$D(p, q) < f_1(p) \Rightarrow f(q) < 2f(p).$$

It follows from Theorem 6.2 that there is a null sequence of mutually exclusive disks  $E_1, E_2, \dots$  and a monotone map  $g$  of  $M^2$  onto itself such that

$$M^2 - \sum \text{Int } E_i \text{ lies on a tame 2-manifold in } M^3,$$

$$\text{diameter } E_i < \min_{f_1} E_i,$$

$$D(m, g(m)) < f(m)/5,$$

$$g^{-1} \text{ is either an } E_i \text{ or a point of } M^2 - \sum E_i.$$

Let  $f_2$  be a continuous function defined on  $M^2$  such that

$$D(g(p), g(a)) < f_2 g(a) \Rightarrow f(p) < 2f(a).$$

Also, let  $f_3$  be a continuous positive function defined on  $M^2$  such that

$$f_3 g(p) < f(p)/5.$$

Consider a triangulation  $T'$  of  $M^2$  such that the 1-skeleton of  $T'$  misses each  $g(E_i)$  and for each simplex  $s$  of  $T'$

diameter of  $s$  is less than either  $\min_{f_2}(s)$  or  $\min_{f_3}(s)$ .

Let  $T$  be the triangulation of  $M^2$  such that for each simplex  $s$  of  $T'$ ,  $g^{-1}(s)$  is a simplex of  $T$ . To see that  $\text{diameter } g^{-1}(s) < \min_f g^{-1}(s)$ , consider three points  $p, q, a$  of  $g^{-1}(s)$  such that  $D(p, q) = \text{diameter } g^{-1}(s)$ ,  $f(a) = \min_f g^{-1}(s)$ . Then

$$\begin{aligned} \text{diameter } g^{-1}(s) &= D(p, q) < f(p)/5 + f(q)/5 + D(g(p), g(q)) \\ &< f(p)/5 + f(q)/5 + f_3 g(a) \leq 2f(a)/5 + 2f(a)/5 + f(a)/5 \\ &= f(a) = \min_f g^{-1}(s). \end{aligned}$$

**THEOREM 6.4.** *Suppose  $T_1$  is a triangulation of a 2-manifold  $M^2$  embedded in a 3-manifold  $M^3$  so that the 1-skeleton of  $T_1$  is tame. Then for each positive continuous function  $f$  defined on  $M^2$  there is a triangulation  $T_2$  of  $M^2$  refining  $T_1$  such that the 1-skeleton of  $T_2$  is tame and for each simplex  $\sigma$  of  $T_2$ ,  $\text{diameter } \sigma < \min_f(\sigma)$ .*

**Proof.** We pick  $f_1, \{E_i\}, g, f_2, f_3$  as in the proof of Theorem 6.3 but with the added precaution that the  $E_i$ 's miss the 1-skeleton of  $T_1$  (as we picked the  $E_i$ 's to miss  $\sum \text{Bd } F_i$  in the proof of Theorem 6.1) and in defining  $g$  we pick it to be the identity on the 1-skeleton of  $T_1$  (as we picked  $g$  to be the identity on  $\sum \text{Bd } F_i$  in the proof of Theorem 6.2). The triangulation  $T'$  is taken to refine  $T_1$  so that the 1-skeleton of  $T'$  misses the  $g(E_i)$ 's and each simplex  $s$  of  $T'$  is of diameter less than the minimum value of  $\min_{f_2}(s)$  or  $\min_{f_3}(s)$ . The simplexes of  $T_2$  are the  $g^{-1}(s)$ 's.

**7. Maintaining separation.** We finally give the theorem which shows that the tame 2-manifold  $h_0(M)$  constructed in the proof of Theorem 1.1 is two-sided.

**THEOREM 7.1.** *Suppose  $X$  is a subset of  $E^n$  that separates the point  $p$  from the point  $q$ ,  $U$  is a bounded subset of  $X$ , and  $h_t$  ( $0 \leq t \leq 1$ ) is a homotopy of  $X \rightarrow E^n - (\{p\} + \{q\})$  such that  $h_0$  is the identity and  $h_1$  is fixed on  $X - U$ . Then  $h_1(X)$  separates  $p$  from  $q$ .*

**Proof.** We suppose that  $X$  is closed since if a set separates two points of  $E^n$ , some closed set in it does.

We alter  $E^n$  and  $X$  by adding the point at infinity to each. This makes  $X$  compact. We then recover  $E^n$  by deleting the point  $q$ . We now have that  $p$  belongs to a bounded component of  $E^n - X$  and wish to show that  $p$  belongs to a bounded component of  $E^n - h_1(X)$ .

Let  $S^{n-1}$  be the unit  $(n-1)$ -sphere with center at  $p$  and let  $r$  be the retraction of  $E^n - \{p\}$  onto  $S^{n-1}$  that takes open rays from  $p$  onto the

point where they pierce  $S^{n-1}$ . It follows from Theorem VI 10 on page 97 of [9] that  $r: X \rightarrow S^{n-1}$  is not homotopic to a constant map. It also follows from this same Theorem VI 10 that  $p$  belongs to a bounded component of  $E^n - h_1(X)$  unless  $r: h_1(X) \rightarrow S^{n-1}$  is homotopic to a constant map.

Assume that  $p$  does not belong to a bounded component of  $E^n - h_1(X)$  and  $f_t$  ( $0 \leq t \leq 1$ ) is a homotopy of  $h_1(X) \rightarrow S^{n-1}$  such that  $f_0 = r$  and  $f_1 = \text{constant map}$ . Then

$$g_t = \begin{cases} rh_{2t} & (0 \leq t \leq 1/2), \\ f_{2t-1}h_1 & (1/2 \leq t \leq 1) \end{cases}$$

shows that  $r: X \rightarrow S^{n-1}$  is homotopic to a constant map since  $g_0 = r$  and  $g_1 = \text{constant map}$ . The assumption that  $p$  does not belong to a bounded component of  $E^n - h_1(X)$  led to this contradiction.

**COROLLARY 7.2.** *Suppose  $M^2$  is a 2-manifold embedded as a closed set in a 3-manifold  $M^3$ ,  $U$  is an open subset of  $M^3$  homeomorphic to  $E^3$ ,  $X$  is a compact subset of  $M^2$ .  $U$  and  $h$  is a homeomorphism of  $M^2$  into  $M^3$  such that  $h$  is fixed on  $M^2 - X$  and  $H(X) \subset U$ . Then  $h(M^2)$  separates  $M^3$  if and only if  $M^2$  does. If  $p, q$  are two points of  $M^3 - U$  separated in  $M^3$  by  $M^2$ , they are also separated in  $M^3$  by  $h(M^2)$ .*

**QUESTION.** Can Theorem 7.1 be generalized by replacing  $E^n$  by an  $n$ -manifold where it is understood that  $U$  has a compact closure?

**8. Corrections to [5].** While studying [5], Stephen Slack pointed out some minor errors in it. Although the proof given in [5] contains some simplifications over the proof given in [2], three of the figures used in [5] do not reflect these simplifications and are more applicable to [2]. For example, the assumption for the *very, very special case* (pp. 151-154 of [5]) is so strong that no component of  $S - R_2$  has a triod in its boundary. Hence, the situation could not be as bad as depicted in Figure 3 of [5]. Since each component of  $S \cdot R_2$  intersects  $R_1$ , Figure 5 is inapplicable. Also, the assumption for the *very special case* is enough to prevent anything as bad as that shown in Figure 7.

The proof of the *very special case* (pp. 154-161 of [5]) overlooked the fact that two  $X_i$ 's as defined on page 155 might intersect in two points and the corresponding  $K(X_i)$ 's as defined on page 156 might intersect in a nondegenerate subcontinuum of each. This is remedied if one replaces the definition of  $K(X_i)$  on page 156 with the following.

"For each  $X_i$  that does not share two points with any preceding  $X_i$ , let  $K(X_i)$  be the sum of  $X_i$  and all components of  $S - X_i$  with diameters less than  $\epsilon/2$ . Other  $K$ 's are defined inductively. If  $X_i$  shares two points with some preceding  $X$ 's and the  $K$ 's on these preceding  $X$ 's have been defined, let  $X'_i$  be the closure of  $X_i$  minus the  $K$ 's associated with these

preceding  $X$ 's and let  $K(X_i)$  be the sum of  $X_i$  and all components of  $S - X_i$  with diameters less than  $\epsilon/2$ ."

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